A NOTE ON THE CARTESIAN FORMULATION OF THE MEMBRANE THEORY

by

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SYNOPSIS

A method of determining the stresses in a membrane shell that is subjected to boundary conditions of the displacement or elastic type is presented without consideration of the displacements. The method is formulated in cartesian coordinates and is based on the principle of stationary complementary energy together with the use of Pucher's stress function.

The same principle is also used to join to the differential equation for the stress function F a second differential equation for a Lagrange multiplier λ and to derive the 2 boundary conditions for F and λ that are equivalent to 2 given boundary conditions of the displacement or elastic type.

A formulation in cartesian coordinates of the stress-strain-displacement relations of the membrane theory is presented and is found to agree with results stated elsewhere.

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Introduction

The equilibrium equations of the membrane theory in cartesian coordinates are usually reduced, following Pucher's approach, to one second order differential equation for a stress function[2] F. The determination of the stress resultants by solving the differential equation and without consideration of the displacements is possible only if one force boundary condition is imposed.

In actual shell problems, however, all or part of the boundary conditions may be of the displacement and possibly of the elastic type. In this case there is no given boundary condition for the stress function and a direct determination of the stress resultants does not seem feasible without first determining the displacements. It seems therefore desirable especially when the displacements are not sought to devise a method for a direct determination of the stress resultants.

This method consists in the application of the principle of stationary complementary energy in a way that satisfies the boundary conditions of the problem.

When the use of the direct methods of the calculus of variations is not appropriate, a recourse to the solution of differential equations is necessary. The principle of stationary complementary energy is used to derive a second differential equation that has to be solved with the differential equation for the stress function. The boundary conditions for the 2 differential equations are also established.

The derivation of stress strain relations and of strain-displacement relations from the expression of the complementary energy and through the use of the theorem of virtual work, respectively, is an established procedure in the theory of elasticity and in shell theory[3,4]. It is performed here in cartesian coordinates and is found to agree with earlier results.[3,5]

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2. Timoshenko and Woinowsky-Kreiger, "Theory of Plates and Shells", 2nd edition, McGraw Hill, 1959, p. 461.

3. Reissner, Eric, "On Some Aspects of the Theory of Thin Elastic Shells", J. Boston Soc., Civ. Eng. 1955, p. 100.

Differential Equations of Equilibrium

The equilibrium equations of a membrane in cartesian coordinates are conveniently formulated in terms of projected stress resultants N_{xx}, N_{xy}, N_{yx}, N_{yy}, Fig. 1.

These are related to the actual stress resultants through the relations

$$N_{xx} = \left(\frac{1+Z_{,y}^2}{1+Z_{,x}^2}\right)^{\frac{1}{2}} N_{11} \quad (1) \qquad\qquad N_{yy} = \left(\frac{1+Z_{,x}^2}{1+Z_{,y}^2}\right)^{\frac{1}{2}} N_{22} \quad (2)$$

$$N_{xy} = N_{12} \quad (3) \qquad\qquad N_{yx} = N_{21} \quad (4)$$

where $Z = Z(x, y)$ is the equation of the middle surface.

The force equilibrium equations may be written in the form

$$N_{xx,x} + N_{yx,y} + p_x = 0 \tag{5}$$

$$N_{xy,x} + N_{yy,y} + p_y = 0 \tag{6}$$

$$(Z,_x N_{xx} + Z,_y N_{xy}),_x + (Z,_x N_{yx} + Z,_y N_{yy}),_y + p_z = 0 \tag{7}$$

and the moment equation yields

$$N_{xy} = N_{yx} \tag{8}$$

At the boundary the projected stress resultants defined with regard to the normal direction n and the tangential direction s are related to N_{xx}, N_{yy}, and N_{xy} through the transformation formulas

$$N_{nn} = N_{xx} \cos^2\varphi + 2N_{xy} \sin\varphi \cos\varphi + N_{yy} \sin^2\varphi \tag{9}$$

$$N_{ss} = N_{xx} \sin^2\varphi - 2N_{xy} \sin\varphi \cos\varphi + N_{yy} \cos^2\varphi \tag{10}$$

$$N_{ns} = N_{xy} (\cos^2\varphi - \sin^2\varphi) + (N_{yy} - N_{xx}) \sin\varphi \cos\varphi \tag{11}$$

where φ is the angle between the x axis and the oriented normal, Fig. 2.

The membrane equilibrium equations are satisfied by letting

$$N_{xx} = F,_{yy} - J_x \tag{12}$$

$$N_{yy} = F,_{xx} - J_y \tag{13}$$

$$N_{xy} = N_{yx} = - F,_{xy} \tag{14}$$

and requiring the stress function F to satisfy the differential equation

$$L(F) = Z,_{xx} F,_{yy} - 2Z,_{xy} F,_{xy} + Z,_{yy} F,_{xx} = q \tag{15}$$

In eqs. 5 to 7, p_x, p_y and p_z denote the components of the surface load per unit projected area in the x, y plane. J_x, J_y and q are defined as follows

$$J_x = \int_{x_o}^{x} p_x \, dx \tag{16}$$

$$J_y = \int_{y_o}^{y} p_y \, d_y \tag{17}$$

$$q = - p_z + Z,_x p_x + Z,_y p_y + Z,_{xx} J_x + Z,_{yy} J_y \tag{18}$$

The limits of integrations x_o and y_o must be independent of x and y, respectively, but are otherwise arbitrary.

The solvability of eq. 15 as depending on the gaussian curvature of the middle surface and on the type of boundary conditions is outside the scope of this paper.[5, 6] It will be assumed in the following that the problem at hand does have a solution and the principle of stationary complementary energy will be used to obtain such a solution when displacement boundary conditions are imposed on all or part of the boundary

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4. Reissner, Eric, "Variational Considerations for Elastic Beams and Shells", Journal of the Engineering Mechanics Division, ASCE, 1962; Proc. Paper No. 3057.

5. Flugge, W. and Geyling, F., "A General Theory of Deformation of Membrane Shells" IXe Congres International de Mecanique Appliquee, Actes, Tome vi, Universite de Bruxelles, 1957, p. 250.

6. Beschkine, Leon, "Sur les Equations d'equilibre des Surfaces minces" Comptes Rendus des Seances del'Academie des Sciences, Paris, Gauthier-Villars, 1935, p. 935.

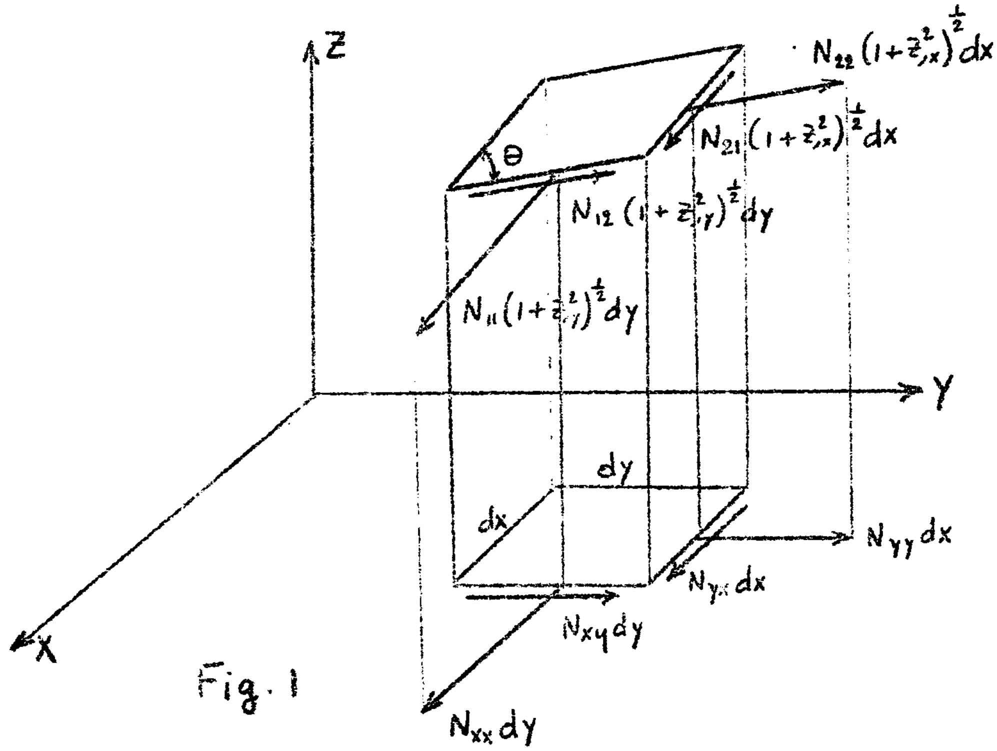
For a membrane 2 displacement boundary conditions may be specified. They usually involve the tangential displacements only. The reason for this is that a specification of the component of displacement normal to the shell or of the rotation at the boundary makes the membrane solution generally unrelated to the state of stress away from the boundary of the shell. This may also be established by introducing the appropriate force constraints into a general variational theorem of thin shell theory that may be found in ref. 4 and noticing that the effect of the constraints on the natural boundary conditions of the variational equation is to alter those specifying the normal displacement and the rotation but to leave unchanged those specifying the tangential components of displacement.

Because the force formulation is not made in terms of the actual stress resultants but in terms of projected stress resultants, it is more convenient to specify the displacement boundary conditions in a form that does not involve the tangential displacements themselves but 2 other displacement components u_n' and u_s'. These are related to the actual tangential displacements by requiring the expression $N_{nn} u_n' + N_{ns} u_s'$ to represent the work per unit projected length of the membrane forces through the displacements at the boundary. N_{nn} and N_{ns} are the normal and tangential projected stress resultants.

The boundary conditions may be taken in the form[4]

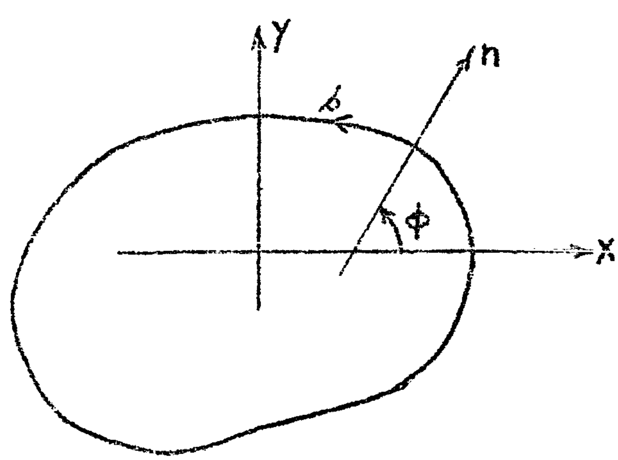
$$u_n' = -\frac{\partial \psi}{\partial N_{nn}} \qquad (19) \qquad\qquad u_s' = -\frac{\partial \psi}{\partial N_{ns}} \qquad (20)$$

where ψ is a known function of N_{nn} and N_{ns} at the boundary. For specified boundary values of u_n' and u_s' or for elastic boundary conditions expressing u_n' and u_s' as functions of N_{nn} and N_{ns}, ψ can readily be constructed. In particular for a clamped boundary the membrane boundary conditions are $u_n' = u_s' = 0$ and ψ may be taken identically zero. In general ψ may be expressed in terms of derivatives of the stress function through the relations



Z
$N_{22}(1+z_{,x}^2)^{\frac{1}{2}}dx$
$N_{21}(1+z_{,x}^2)^{\frac{1}{2}}dx$
θ
$N_{12}(1+z_{,y}^2)^{\frac{1}{2}}dy$
$N_{11}(1+z_{,y}^2)^{\frac{1}{2}}dy$
Y
dy
dx
$N_{yy}dx$
$N_{yx}dx$
$N_{xy}dy$
X
$N_{xx}dy$

Fig. 1



Y
n
b
φ
X

Fig. 2

$$N_{nn} = F_{,ss} + \quad_{,s} F_{,n} - J_x \cos^2\varphi - J_y \sin^2\varphi \tag{21}$$

$$N_{ns} = - F_{,ns} + \varphi_{,s} F_{,s} + (J_x - J_y) \sin\varphi \cos\varphi \tag{22}$$

where a comma followed by s or n indicates differentiation with regard to the arclength of or normal to the projected boundary, respectively. It is noted in that respect that the meaning of $F_{,ns} = \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial n}\right)$ is clear, contrary to $F_{,sn}$ which requires $F_{,s}$ to be defined not only at the boundary but also in its neighborhood. In fact $F_{,sn} = F_{,ns} - \varphi_{,s} F_{,s}$.

The Complementary Strain Energy

In terms of stress resultants N'_{ij} defined with regard to orthogonal curvilinear coordinates and for a linearly elastic and isotropic material the complementary strain energy reduces to the strain energy expressed in terms of the stress resultants and takes the form

$$W = \frac{1}{2Eh}\left[(N'_{11} + N'_{22})^2 - 2(1+\nu)(N'_{11} N'_{22} - (N'_{12})^2\right]$$

In order to express W in terms of the projected stress resultants, N'_{ij} may first be expressed in terms of the stress resultants N_{ij} defined with regard to the non-orthogonal curvilinear coordinates x and y. For this purpose the directions on the surface corresponding to the subscript 1 are taken as the same. Letting θ denote the angle between the coordinate lines (x) and (y) on the surface, Fig. 1, N'_{ij} and N_{ij} are related through the relations

$$N'_{11} = \frac{1}{\sin\theta}\left[N_{11} + N_{22} \cos^2\theta + (N_{12} + N_{21}) \cos\theta\right]$$

$$N'_{22} = N_{22} \sin\theta$$

$$N'_{12} = N_{12} + N_{22} \cos\theta$$

$$N'_{21} = N_{21} + N_{22} \cos\theta$$

where

$$\cos\theta = Z,_x\, Z,_y\, (1 + Z,_x^2)^{-1/2}\, (1 + Z,_y^2)^{-1/2}$$

$$\sin\theta = (1 + Z,_x^2 + Z,_y^2)^{1/2}\, (1 + Z,_x^2)^{-1/2}\, (1 + Z,_y^2)^{-1/2})$$

W takes the form

$$W = \frac{1}{2Eh\sin^2\theta}\left[(N_{11} + N_{22} + 2N_{12}\cos\theta)^2 - 2(1+\nu)\sin^2\theta(N_{11}N_{22} - N_{12}^2)\right]$$

In terms of the projected stress resultants, eqs. 1 to 4, and per unit area in the x, y plane the complementary strain energy takes the form

$$W* = \frac{(1 + Z,_x^2 + Z,_y^2)^{-1/2}}{2Eh}\left[\left[(1 + Z,_x^2)N_{xx} + (1 + Z,_y^2)N_{yy} + 2Z,_x Z,_y N_{xy}\right]^2 - 2(1+\nu)(1 + Z,_x^2 + Z,_y^2)(N_{xx}N_{yy} - N_{xy}^2)\right] \quad (13)$$

Variational Formulation

A functional having as Euler equations and natural boundary conditions the equilibrium equations, force-displacement relations and boundary conditions of thin elastic shells is established in reference 4. The argument functions of the functional are displacements, stress resultants, and stress couples. In order to be used in the present problem the functional must first be specialized for the membrane theory but it must also be modified so that only variations of forces are considered. This latter modification leads to the principle of stationary complementary energy.

The functional takes then the form

$$E = \iint W*\, dx\, dy + \int_D \Psi\, ds$$

where the first integral extends over the projected middle surface and the second integral extends over the portion D of the projected boundary curve where displacement boundary conditions are specified. The stress function F makes stationary the functional E with 2 restrictions that the varied function $F + \delta F$ satisfies the equilibrium equation, (15), and possibly a stress boundary condition which may be specified on a portion S of the boundary. Thus the restriction on δF in the domain of integration takes the form of the subsidiary condition

$$\delta L(F) = \delta(Z,_{xx} F,_{yy} - 2Z,_{xy} F,_{xy} + Z,_{yy} F,_{xx}) = 0 \qquad (24)$$

which can be treated by introducing a Lagrange multiplier[7], λ, function of x and y. The functional takes the form

$$E' = \iint (W^* + \lambda L(F - q))dx\, dy + \int_D \psi ds \qquad (25)$$

The variation δF of F is now arbitrary throughout the domain of integration and on the portion D of the boundary. The only restriction is that $F + \delta F$ satisfies the stress boundary condition, if any, on S.

If the general solution of the differential equation for the stress function is known, it will depend on arbitrary constants or functions of integration that may be arbitrarily varied without violating condition (24). In that case there is no need to introduce the Lagrange multiplier and the application of the direct methods of the calculus of variations yields the appropriate equations for the determination of the arbitrary quantities in the expression of the stress function. An illustrative example is presented in a subsequent paragraph. If the general solution for the stress function cannot be determined analytically, approximate methods for expressing the stress function in terms of sufficiently many arbitrary quantities may be used. One such method consists in replacing the differential equation $L(F) - q = 0$ by a finite difference equation at each of n points in the interior of the membrane. This yields a system of n linear algebraic equations in n + m unknowns of which n are the values of F at the n interior points and m are the values of F at m boundary points. These latter are arbitrary constants on which the values of F in the interior of the membrane depend. The functional E which may be computed by numerical integration is thus transformed into a function of m variables with regard to which it is made stationary.

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7. Courant and Hilbert, "Methods of Mathematical Physics", Vol. 1, Interscience Publishers, Inc., New York, 1953, p. 221.

Another possible method of expressing F in terms of arbitrary constants consists in determining an approximate continuous solution of the differential equation L)F) - q = 0 in terms of arbitrary boundary values of F. These may be specified through a continuous function of position along the boundary depending on undetermined constants. When it is not advantageous to use direct methods such as described above the problem of the membrane with displacement boundary conditions may be reduced to the solution of eq. 15 and of the Euler equation with the natural boundary conditions of the functional E', eq. 25. These are derived in the next paragraph.

Euler Equation and Natural Boundary Conditions

The first variation of E' can be written in the form

$$\delta E' = \iint\left[(\varepsilon_{xx}+\lambda z_{,xx})\delta F_{,yy} - 2(\varepsilon_{xy}+\lambda z_{,xy})\delta F_{,xy} + (\varepsilon_{yy}+\lambda z_{,yy})\delta F_{,xx}\right]dx\,dy$$

$$+\int_D \left[\frac{\partial \Psi}{\partial N_{nn}}\delta(F_{,ss}+\varphi_{,s}F_{,n}) + \frac{\partial \Psi}{\partial N_{ns}}\delta(-F_{,ns}+\varphi_{,s}F_{,s})\right]ds \tag{26}$$

where, using eq. 23 and adopting the notation ε_{xx}, $\varepsilon_{,yy}$, $2\varepsilon_{xy}$, for the partial derivatives of W^*,

$$\varepsilon_{xx} = \frac{\partial W^*}{\partial N_{xx}} = \frac{(1+z_{,x}^2+z_{,y}^2)^{-\frac{1}{2}}}{Eh}\Big[(1+z_{,x}^2)\left[(1+z_{,x}^2)N_{xx}+(1+z_{,y}^2)N_{yy}+2z_{,x}z_{,y}N_{xy}\right]$$
$$-(1+\nu)(1+z_{,x}^2+z_{,y}^2)N_{yy}\Big] \tag{27}$$

$$\varepsilon_{yy} = \frac{\partial W^*}{\partial N_{yy}} = \frac{(1+z_{,x}^2+z_{,y}^2)^{-\frac{1}{2}}}{Eh}\Big[(1+z_{,y}^2)\left[(1+z_{,x}^2)N_{xx}+(1+z_{,y}^2)N_{yy}+2z_{,x}z_{,y}N_{xy}\right]$$
$$-(1+\nu)(1+z_{,x}^2+z_{,y}^2)N_{xx}\Big] \tag{28}$$

$$2\varepsilon_{xy} = \frac{\partial W^*}{\partial N_{xy}} = \frac{2(1+z_{,x}^2+z_{,y}^2)^{-\frac{1}{2}}}{Eh}\Big[z_{,x}z_{,y}\left[(1+z_{,x}^2)N_{xx}+(1+z_{,y}^2)N_{yy}+2z_{,x}z_{,y}N_{xy}\right]$$
$$+(1+\nu)(1+z_{,x}^2+z_{,y}^2)N_{xy}\Big] \tag{29}$$

Integration by parts must now be used so that only δF appears in the area integral and only δF and $\delta F_{,n}$ appear on the boundary integral. Using Green's Theorem the following general relation may be established

$$\iint (f_{xx}\delta F_{,yy} - 2f_{xy}\delta F_{,xy} + f_{yy}\delta F_{,xx})\,dx\,dy = \iint (f_{xx,yy} - 2f_{xy,xy} + f_{yy,xx})\delta F\,dx\,dy$$

$$+ \oint f_{ss}\delta F_{,n}\,ds - \oint \left[f_{ss,n} - 2f_{ns,s} + \varphi_{,s}(f_{ss} - f_{nn})\right]\delta F\,ds \tag{30}$$

where f_{xx}, f_{xy}, and f_{yy} are 3 functions of x and y and f_{nn}, f_{ss} and f_{ns} are obtained through the same transformation formulas as eqs. 9 to 11.

$$f_{nn} = f_{xx}\cos^2\varphi + 2f_{xy}\sin\varphi\cos\varphi + f_{yy}\sin^2\varphi \tag{31}$$

$$f_{ss} = f_{xx}\sin^2\varphi - 2f_{xy}\sin\varphi\cos\varphi + f_{yy}\cos^2\varphi \tag{32}$$

$$f_{ns} = f_{xy}(\cos^2\varphi - \sin^2\varphi) + (F_{yy} - f_{xx})\sin\varphi\cos\varphi \tag{33}$$

Identifying f_{xx}, $-2f_{xy}$ and f_{yy} with the coefficients of $\delta F_{,yy}$, $\delta F_{,xy}$ and $\delta F_{,xx}$, respectively, in eq. 26 the Euler differential equation is obtained by equating to zero the coefficient of δF in the area integral on the right of eq. 30. Obtain after simplification

$$L(\lambda) + \varepsilon_{xx,yy} - 2\varepsilon_{xy,xy} + \varepsilon_{yy,xx} = 0 \tag{34}$$

Eqs. 34 and 15 i. e.

$$L(F) = q \tag{35}$$

form 2 equations for F and the Lagrange multiplier λ. ε_{xx}, ε_{xy} and ε_{yy} are expressed in terms of F using eqs. 27 to 29 and eqs. 12 to 14.

In order to obtain the natural boundary conditions integration by parts is performed on the boundary integral in eq. 26 and the result is added to the portion of the boundary integrals in eq. 30 taken over the same domain D. Then the coefficients of δF and $\delta F_{,n}$ put to zero form the natural boundary conditions. Obtain

$$\varepsilon_{ss} + \lambda(z_{,ss} + \varphi_{,s}z_{,n}) + \left(\frac{\partial\psi}{\partial N_{ns}}\right)_{,s} + \varphi_{,s}\frac{\partial\psi}{\partial N_{nn}} = 0 \tag{36}$$

$$\left[\varepsilon_{ss} + \lambda(z_{,ss} + \varphi_{,s}z_{,n})\right]_{,n} - 2\left[\varepsilon_{ns} + \lambda(z_{,ns} - \varphi_{,s}z_{,s})\right]_{,s}$$

$$- \varphi_{,s}(\varepsilon_{nn} + \lambda z_{,nn}) - \left(\frac{\partial\psi}{\partial N_{nn}}\right)_{,ss} - \varphi_{,s}^2\frac{\partial\psi}{\partial N_{nn}} + \varphi_{,ss}\frac{\partial\psi}{\partial N_{ns}} = 0 \tag{37}$$

where ε_{nn}, ε_{ns} and ε_{ss} are related to ε_{xx}, ε_{xy} and ε_{yy} through transformation formulas similar to eqs. 31 to 33.

For eqs. 36 and 37 to form effectively a system of 2 boundary conditions $(Z,_{ss} + \varphi,_s Z,_n)$ must be non zero, that is, the normal curvature of the surface in the direction tangent to the boundary curve must not be zero. This is always the case for a surface of positive gaussian curvature. However, if the surface has a negative gaussian curvature and part of the boundary curve coincides with an asymptotic line eq. 36 does not contain λ and becomes a boundary condition for F. The problem then is not properly formulated as the differential equation for F is of the hyperbolic type.

If there is a force boundary condition taking for example the form of an assigned value of the stress function on a part (S) of the boundary, the variation δF must vanish on (S) but $\delta F,_n$ remains arbitrary. Only one natural boundary condition is then obtained on (S), namely eq. 36 which is a boundary condition for λ. A case of interest[5,8] is the boundary condition F = 0 with one displacement boundary condition $u'_s = 0$. Eq. 36 yields then

$$\lambda = -\frac{\varepsilon_{ss}}{Z,_{ss} + \varphi,_s Z,_n} \tag{38}$$

on the part (S) of the boundary.

It will be shown subsequently that λ is identical with the component of displacement along the Z axis.

Application

Consider a shallow paraboloid of revolution with a rectangular projected boundary, such as shown in Fig. 3, and subjected to a uniform vertical load p.

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8. V. V. Novozhilov, "Thin Shell Theory", 2nd Edition, P. Noordhoff Ltd., Groningen, 1964, p. 182.

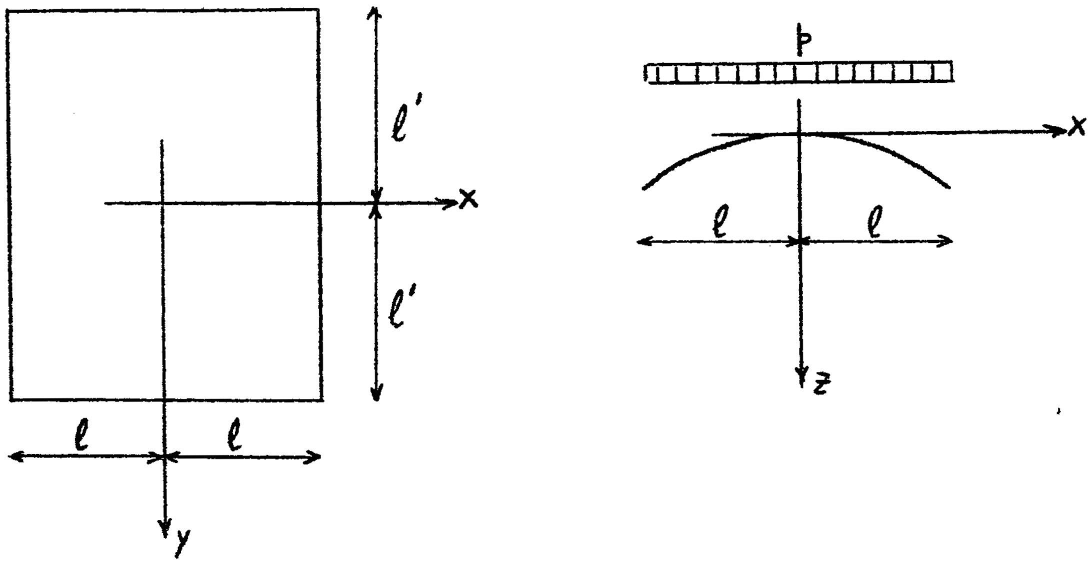

Figure 3

The edges at $x = \pm \ell$ are joined to beams assumed to be infinitely stiff in their own plane but infinitely flexible perpendicularly to their plane. Such boundary conditions require

$$N_{xx} = 0 \quad , \quad u'_s = 0 \quad \text{at } x = \pm \ell$$

The condition $N_{xx} = 0$ may be reduced to

$$F = \text{constant} \qquad \text{at } x = \pm \ell \tag{39}$$

The edges at $y = \pm \ell'$ are assumed to be fixed. The corresponding membrane boundary conditions are

$$u'_s = u'_n = 0 \qquad \text{at } y = \pm \ell' \qquad \text{or equivalently}$$

$$\psi = 0 \qquad \text{at } y = \pm \ell' \tag{40}$$

The equation of the middle surface may be taken in the form

$$Z = \frac{x^2 + y^2}{2a} \tag{41}$$

whence the equation for the stress function

$$F,_{yy} + F,_{xx} = - ap \tag{42}$$

A particular solution of eq. (42) that satisfies the boundary condition (39) may be taken in the form

$$F_p = - ap \frac{x^2}{2} \tag{43}$$

to which correspond the stress resultants

$$N_{xx} = F,_{yy} = 0$$

$$N_{yy} = F,_{xx} = - ap$$

$$N_{xy} = - F,_{xy} = 0$$

The general solution F_h of the homogeneous equation is a harmonic function which in our case must identically satisfy the condition F = constant at $x = \pm \ell$. The constant value of F_h at $x = \pm \ell$ may without loss of generality be taken as zero. A suitable form for F_h is

$$F_h = \sum_{n=0}^{\infty} G_n (y) \cos \frac{(2n + 1)\, \pi x}{2\ell} \tag{44}$$

Due to symmetry about the (y, z) plane only cosine terms are considered. These vanish for all values of n at $x = \pm \ell$. Substituting for F_h into the differential equation and requiring it to be satisfied for the general term in the expression of F_h yields the equation

$$G_{n,yy} - \frac{(2n + 1)^2 \pi^2}{4\, \ell^2}\, G_n = 0 \tag{45}$$

The general solution of which after enforcing symmetry may be written in the form

$$G_n = C_n \cosh \frac{(2n + 1)\, \pi y}{2\ell} \tag{46}$$

where C_n is an arbitrary constant of integration. F takes the form

$$F = - ap \frac{x^2}{2} + \sum_{n=0}^{\infty} C_n \cosh \frac{(2n + 1)\pi y}{2\ell} \cos \frac{(2n + 1)\, \pi x}{2\ell} \tag{47}$$

With the assumption of shallowness i. e.

$$Z,_x^2 = \frac{x^2}{a^2} \ll 1$$

$$Z,_y^2 = \frac{y^2}{a^2} \ll 1$$

and noting that

$$F_{h,yy} + F_{h,xx} = 0$$

Eqs. 27 to 29 reduce to

$$\varepsilon_{xx} = \frac{1}{Eh}(N_{xx} - \nu N_{yy}) = \frac{\nu ap}{Eh} + \frac{1+\nu}{Eh} F,_{yy} \tag{48}$$

$$\varepsilon_{yy} = \frac{1}{Eh}(N_{yy} - \nu N_{xx}) = -\frac{ap}{Eh} - \frac{1+\nu}{Eh} F,_{yy} \tag{49}$$

$$2\varepsilon_{xy} = \frac{2(1+\nu)}{Eh} N_{xy} = -\frac{2(1+\nu)}{Eh} F,_{xy} \tag{50}$$

and the variational equation reduces to

$$\int_{-\ell'}^{\ell'} dy \int_{-\ell}^{\ell} \left[(ap + 2F,_{yy})\,\delta F,_{yy} + 2F,_{xy}\,\delta F,_{xy}\right] dx = 0 \tag{51}$$

where $\delta F,_{yy}$, $\delta F,_{xy}$ are the derivatives of

$$\delta F = \sum_{n=0}^{\infty} \delta C_n \cosh \frac{(2n+1)\pi y}{2\ell} \cos \frac{(2n+1)\pi x}{2\ell} \tag{52}$$

and δC_n is an arbitrary variation of C_n.

The evaluation of the terms and the integrations in the variational equation are straight forward. In addition, the orthogonality of the cosine and sine functions in the domain of integration allows the determination of the integration constants one by one. The result is

$$C_n = (-1)^{n+1} \frac{8ap\,\ell^2}{(2n+1)^3 \pi^3 \cosh \frac{(2n+1)\pi \ell'}{2\ell}} \tag{53}$$

whence

$$N_{xx} = F,_{yy} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2ap \cosh \frac{(2n+1)\pi y}{2\ell} \cos \frac{(2n+1)\pi x}{2\ell}}{(2n+1)\pi \cosh \frac{(2n+1)\pi \ell'}{2\ell}} \quad (54)$$

$$N_{yy} = F,_{xx} = - ap - N_{xx} \quad (55)$$

$$N_{xy} = - F,_{xy} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2ap \sinh \frac{(2n+1)\pi y}{2\ell} \sin \frac{(2n+1)\pi x}{2\ell}}{(2n+1)\pi \cosh \frac{(2n+1)\pi \ell'}{2\ell}} \quad (56)$$

The convergence of the series is satisfactory near the apex of the paraboloid but deteriorates when approaching the edges. At the corners the series for N_{xy} diverges which indicates that the membrane solution does not represent the state of stress in these regions. The same phenomenon occurs when all four edges are supported similarly to the edges at $x = \pm \ell$. The numerical results[9] for this last case give for N_{xx} and N_{xy} twice the values in eqs. 54 and 55, respectively.

It is interesting to note that if the particular solution is taken in the form

$$F_p = - \frac{ap}{4} (x^2 + y^2)$$

yielding

$$N_{xx} = N_{yy} = - \frac{ap}{2}$$

$$N_{xy} = 0$$

the variational equation contains only the homogeneous solution and is satisfied by letting $F_h = 0$. $F_p = - \frac{ap}{4}(x^2 + y^2)$ solves then the problem of the shallow paraboloid fixed all around a boundary of arbitrary shape.

Stress-Strain Displacement Relations

A derivation of stress-displacement relations using the principle of stationary complementary energy with the technique of Lagrange multipliers may be found in reference 3. The same results are stated in reference 5

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9. V. V. Novozhilov, Op. Cit., p. 193.

and are obtained apparently through a different approach. Here the stress-strain relations are established by considering the strains as partial derivatives of the complementary strain energy density function and the strain-displacement relations are obtained through the principle of virtual work. These methods are known in 3 dimensional elasticity and in the usual formulation of shell theory in orthogonal curvilinear coordinates. Applied in the cartesian formulation of the membrane theory they yield the same results found elsewhere.

By analogy to other formulations the components of strain in the cartesian formulation of the membrane theory may be defined without geometric considerations by expressing the increment of complementary strain energy density due to arbitrary increments dN_{xx}, dN_{yy}, dN_{xy} and $dN_{yx} = dN_{xy}$ of the projected stress resultants in the form

$$dW^* = \varepsilon_{xx} dN_{xx} + 2\,\varepsilon_{xy} dN_{xy} + \varepsilon_{yy} dN_{yy} \tag{57}$$

where ε_{xx}, ε_{xy}, and ε_{yy} are the components of strain. This amounts to defining the strains through eqs. 27 to 29. The strain-displacement relations expressing the strains in terms of the cartesian components of displacement u_x, u_y and u_z may now be found by requiring the validity of the principle of virtual work which may be written in the form

$$\iint (N_{xx}\,\varepsilon_{xx} + 2N_{xy}\,\varepsilon_{xy} + N_{yy}\,\varepsilon_{yy})\,dx\,dy - \iint (p_x u_x + p_y u_y + p_z u_z)\,dx\,dy$$

$$- \oint (N_{nn}\,u'_n + N_{ns}\,u'_s)\,ds = 0 \tag{58}$$

In eq. 58 the area integrals represent internal and external virtual work respectively, and the line integral represents the virtual work of the boundary stress resultants. This may serve actually to define u'_n and u'_s. Eq. 58 must hold identically for any system of stress resultants and surface loads in equilibrium. Using the equilibrium equations 5 to 8 to substitute for p_x, p_y, and p_z, then applying Green's theorem and equating to zero the coefficients of N_{xx}, N_{xy}, and N_{yy} in the resulting area integral yields

$$\varepsilon_{xx} = u_{x,x} + Z,_x u_{z,x} \tag{59}$$

$$\varepsilon_{yy} = u_{y,y} + Z,_y u_{z,y} \tag{60}$$

$$2\varepsilon_{xy} = u_{x,y} + u_{y,x} + Z,_x u_{z,y} + Z,_y u_{z,x} \tag{61}$$

In the boundary integral N_{xx}, N_{yy} and N_{xy} are expressed in terms of N_{nn}, N_{ss} and N_{ns} through the transformation formulas 9 to 11 inverted, and u_x and u_y are expressed in terms of components normal and tangential to the projected boundary through the relations

$$u_n = u_x \cos\varphi + u_y \sin\varphi \tag{62}$$

$$u_s = - u_x \sin\varphi + u_y \cos\varphi \tag{63}$$

This yields for u_n' and u_s' the defining relations

$$u_n' = u_n + Z,_n u_z \tag{64}$$

$$u_s' = u_s + Z,_s u_z \tag{65}$$

It may be shown through eqs. 64 and 65, consistently with the interpretation of u_n' and u_s' in eq. 58, that these are the components of the displacement vector with regard to the normal and tangent to the projected boundary when the third axis is the normal to the shell. The strain quantities ε_{xx}, ε_{yy} and $2\,\varepsilon_{xy}$ do not represent, though they are related to, what is normally called extensional and shear strains. They are, however, the quantities to be related to the projected stress resultants.

Interpretation of the Lagrange Multiplier and of the Natural Boundary Condition

The Euler differential equation of the variational equation $\delta E' = 0$ takes now the significance of a "compatibility equation" in terms of the stress resultants. Here by "compatibility equation" is meant the equation obtained from eqs. 59 to 61 by eliminating u_x and u_y. In fact, using eqs. 59 to 61 the following identity in the displacements may be obtained.

$$L(u_z) + \varepsilon_{xx,yy} - 2\,\varepsilon_{xy,xy} + \varepsilon_{yy,xx} = 0 \tag{66}$$

Comparing eq. 66 with eq. 34 it is seen that the Lagrange multiplier λ obeys the same differential equation as u_z. In addition, if, for simplicity, a boundary parallel to the x axis is considered, the boundary condition (38) corresponding to $u'_x = 0$ takes the form

$$\lambda = -\frac{\varepsilon_{xx}}{Z,_{xx}}$$

From eqs. 59 and 65 the equivalent condition $u'_{x,x} = 0$ yields

$$u_z = -\frac{\varepsilon_{xx}}{Z,_{xx}}$$

It appears therefore that the Lagrange multiplier λ may be identified with the displacement component u_z.

In deriving the natural boundary conditions, eqs. 36 and 37, no reference to the displacements was made and the quantities ε_{nn}, ε_{ss} and $2\,\varepsilon_{ns}$ were defined directly in terms of the stress resultants. Now, however, that these are established as components of strain it is interesting to establish the form the natural boundary conditions take when the strains are expressed in terms of the displacement components. The result should obviously be a consequence of the original displacement boundary conditions 19 and 20 and should therefore be expressable in terms of u'_n and u'_s only. In order to carry out this transformation, the tensor character of the components of strain, which is indicated by the transformation formulas eqs. 32 to 33 consistently with eqs. 27 to 29 and eqs. 59 to 61, is used, yielding

$$\varepsilon_{nn} = u_{n,n} + Z,_n\, u_{z,n} \tag{67}$$

$$\varepsilon_{ss} = u_{s,s} + \varphi,_s\, u_n + Z,_s\, u_{z,s} \tag{68}$$

$$2\,\varepsilon_{ns} = u_{n,s} + u_{s,n} - \varphi,_s\, u_s + Z,_s\, u_{z,n} + Z,_n\, u_{z,s} \tag{69}$$

In terms of u'_n and u'_s as defined through eqs. 64 and 65, eqs. 67 to 69 take the form

$$\mathcal{E}_{nn} + Z,_{nn} u_z = u'_{n,n} \tag{70}$$

$$\mathcal{E}_{ss} + (Z,_{ss} + \varphi,_s Z,_n)u_z = u'_{s,s} + \varphi,_s u'_n \tag{71}$$

$$2\,\mathcal{E}_{ns} + 2(Z,_{ns} - \varphi,_s Z,_s)u_z = u'_{n,s} + u'_{n,s} + u'_{s,n} - \varphi,_s u'_s \tag{72}$$

It is seen that the quantities on the left of eqs. 70 to 72 are exactly those occurring in the natural boundary conditions. Upon substitution these take the form

$$u'_{s,s} + \varphi,_x u'_n + (\frac{\partial \Psi}{\partial N_{ns}},_s + \varphi,_s \frac{\partial \Psi}{\partial N_{nn}} = 0 \tag{73}$$

$$u'_{n,ss} + \varphi,_s^2 u'_n - \varphi,_{ss} u'_s + (\frac{\partial \Psi}{N_{nn}}),_{ss} + \varphi,_s^2 \frac{\partial \Psi}{\partial N_{nn}} - \varphi,_{ss} \frac{\partial \Psi}{\partial N_{ns}} = 0 \tag{74}$$

It is noted that eqs. 73 and 74 are indeed a consequence of the boundary conditions in their original form, eqs. 19 and 20. It is interesting to note that the quantities

$$\varepsilon' = u'_{s,s} + \varphi,_s u'_n$$

$$\chi' = u'_{n,ss} + \varphi,_s^2 u'_n - \varphi,_{ss} u'_s$$

occurring on the left of eqs. 72 and 74 may be interpreted as an extensional strain and a change of curvature, respectively, of the projected boundary curve undergoing the plane displacements u'_n and u'_s.

Conclusion

The principle of stationary complementary energy may be used to determine the stress resultants in a membrane subjected to displacement, elastic, or mixed boundary conditions without necessarily determining the displacements.

Alternatively, the determination of the stress resultants maybe performed through the solution of 2 differential equations for the stress function and for the cartesian component of displacement along the Z axis. The functional form of the complementary strain energy density and the principle of virtual work may be used to establish the stress-strain-displacement relations in cartesian coordinates. It seems that the same approach provides a consistent method of formulating in cartesian coordinates the general equations of thin shell theory.

TABLE OF SYMBOLS

E	Young's modulus
E, E'	Functionals
F	Stress function
h	thickness of shell
J_x, J_y	Integrals of load components
L	Operator
n	Subscript associated with the normal to the projected boundary
N	Stress resultants
$p_x, p_y,$ p_z	Components of distributed load intensity
q	Loading term depending on p_x, p_y, p_z and the geometry of the membrane
s	subscript associated with the arclength of the projected boundary
W, W^*	Complementary strain energy density
$u_x, u_y,$ u_z	Components of displacement
x, y, z	Cartesian coordinates
	Subscripts
ε	Strain quantities
φ	Angle between the oriented normal to the projected boundary and the x axis
ψ	Function of the stress resultants acting on the boundary
λ	Lagrange multiplier
ν	Poisson's ratio
θ	Angle between sections of the middle surface by the planes (x, z) and (y, z)

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